B.Math. (Hons.) IInd year
Midsemestral Examination, Semester II 2014
Algebra IV - Instructor : B.Sury
March 3, 2014
Maximum marks : 60

Q 1.
Prove that $X^5 + 12X^3 - 12X + 12$ is irreducible over the field $\mathbf{Q}(e^{2i\pi/7})$.

OR

Determine what the characteristic must be for the polynomial $X^4 + 2X^3 + 3X^2 + 8X + 1$ to have a multiple root.

Q 2.
If f is a monic irreducible polynomial of degree n over $\mathbf{Q}$, show :
(i) the Galois group of f acts transitively on the set of roots of f in a splitting field;
(ii) the discriminant of f is a square in $\mathbf{Q}$ if and only if the Galois group of f consists of even permutations.

OR

Determine the Galois group of the polynomial $X^4 - 2$ over $\mathbf{Q}$. Use this to find the intermediate fields between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[4]{2})$.

Q 3.
If $q = p^n$ and $\alpha \in \mathbf{F}_q$, show that

$$(X - \alpha)(X - \alpha^p)(X - \alpha^{p^2}) \cdots (X - \alpha^{p^{n-1}}) \in \mathbf{F}_p[X].$$

OR

Show that all the irreducible polynomials of degree n over $\mathbf{F}_p$ divide $X^{p^n} - X$ in $\mathbf{F}_p[X]$.

Q 4.
Prove that there exists a Galois extension of $\mathbf{Q}$ whose Galois group is cyclic of order 13.

OR

Let E/F be an extension and let $a \in E$ be algebraic and purely inseparable over F, where char $F = p > 0$. Prove that $\min(F, a) = (X - a)^{p^n}$ for some n.

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Q 5.
Let Char. $K = p > 0$, and let $a \in K$. If the polynomial $X^p - X - a$ is reducible in $K[X]$, prove that all its roots lie in K.

OR

Let L/K be an extension such that each $\alpha \in L$ is algebraic and separable over K with degree at the most d (independent of α). Show that $[L : K] \leq d$.

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Q 6.
Let L/K be a (finite) Galois extension. If the quotient group L^*/K^* contains an element of order n, show that L^* must contain an element of order n.
Hint: If the coset of $a \in L^*$ in L^*/K^* has order n, look at $\sigma(a)/a$ for any σ in $\mathrm{Gal}(L/K)$.

OR

Prove that $\mathbf{Q}(\zeta_n)$ cannot contain a 4-th root of 2 for any n.
Hint: What is the Galois group of $X^4 - 2$?